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# A $p$ -adic analytic approach to the absolute Grothendieck conjecture

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## Introduction

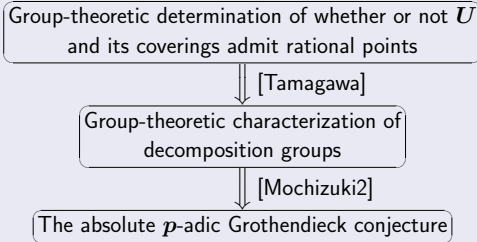
Let  $K$  be a field of characteristic 0,  $\overline{K}$  an algebraic closure of  $K$ ,  $G_K$  the absolute Galois group of  $K$ ,  $U$  a hyperbolic curve over  $K$  and  $\pi_1(U)$  the étale fundamental group of  $U$ . We have the following natural exact sequence:

$$1 \rightarrow \pi_1(U_{\overline{K}}) \rightarrow \pi_1(U) \xrightarrow{\text{pr}} G_K \rightarrow 1.$$

The relative and absolute Grothendieck conjectures ask:

$$\begin{aligned} \text{relative : } & \pi_1(U) \twoheadrightarrow G_K \overset{?}{\underset{\text{recoverable}}{\rightsquigarrow}} U, \\ \text{absolute : } & \pi_1(U) \overset{?}{\underset{\text{recoverable}}{\rightsquigarrow}} U. \end{aligned}$$

When  $K$  is a  $p$ -adic field (i.e. a finite extension of  $\mathbb{Q}_p$ ), the absolute version of Grothendieck conjecture (the absolute  $p$ -adic Grothendieck conjecture) is unsolved. On the other hand, the absolute  $p$ -adic Grothendieck conjecture is reduced to the problem of rational points of curves as follows:



To consider this problem, we shall introduce the “ $i$ -invariant” of **compact** analytic manifold over a  $p$ -adic field  $K$ .

## Theorem-Definition [Serre]

Let  $q$  be the cardinality of the residue field of  $K$  and  $Y$  a nonempty compact analytic manifold over  $K$ . Then  $Y$  is the disjoint union of a finite number of balls and the number of (closed) balls is well determined mod  $(q-1)$ . We call the number of balls  $i_K(Y) \in \mathbb{Z}/(q-1)\mathbb{Z}$  the  $i$ -invariant of  $Y$  over  $K$ . Moreover, we set  $i_K(\emptyset) \equiv 0 \pmod{q-1}$ .

Let  $X$  be a **proper** hyperbolic curve over  $K$  and  $X(K)$  the set of  $K$ -rational points of  $X$ . In terms of the  $i$ -invariants, the absolute  $p$ -adic Grothendieck conjecture (for hyperbolic curves **not necessarily proper**) is reduced to the following two problems:

## Question

- (A) Data of  $i$ -invariants of sets of rational points of  $X$  and its étale coverings  $\rightsquigarrow$  decomposition groups?
- (B) Group-theoreticity of the  $i$ -invariant of  $X(K)$ ?

Theorem A below gives a complete affirmative answer to (A) and Theorem B below gives a partial affirmative answer to (B).

## Theorem A [M]

Let  $K$  be a  $p$ -adic field,  $X$  a proper hyperbolic curve over  $K$  and  $q$  the cardinality of the residue field of  $K$ . Assume that  $q \neq 2$  and let  $m > 1$  be a divisor of  $q-1$ . Then the following conditions are equivalent:

- (i)  $X(K) \neq \emptyset$ .
- (ii)  $\exists X'$ : a finite étale covering of  $X$  such that  $X'(K) \neq \emptyset$ .
- (iii)  $\exists X'$ : a finite étale covering of  $X$  such that  $i_K(X'(K)) \not\equiv 0 \pmod{q-1}$ .
- (iv)  $\exists X'$ : a finite étale covering of  $X$  such that  $i_K(X'(K)) \not\equiv 0 \pmod{m}$ .
- (v)  $\exists X'$ : a finite étale covering of  $X$  such that  $i_K(X'(K)) \equiv (\text{a power of } p) \pmod{q-1}$ .

## Theorem B [M]

For  $i = 1, 2$ , let  $p_i$  be an odd prime number,  $K_i$  a  $p_i$ -adic field and  $X_i$  a proper hyperbolic curve over  $K_i$  which has log smooth reduction (i.e. has stable reduction after tame base extension). Suppose that we are given an isomorphism of profinite groups  $\alpha : \pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)$ . Then we have

$$i_{K_1}(X_1(K_1)) \equiv i_{K_2}(X_2(K_2)) \pmod{2}.$$

In fact, by [Mochizuki1], if we are given an isomorphism  $\alpha : \pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)$ , we have  $p_1 = p_2$  (more precisely, we have  $q_1 = q_2$  where  $q_i$  is the cardinality of the residue field of  $K_i$ ). Moreover,  $X_1$  has log smooth reduction if and only if  $X_2$  has log smooth reduction.

## Remark

If we prove Theorem B without assuming that  $X_i$  has log smooth reduction, we can prove the absolute  $p$ -adic Grothendieck conjecture for  $p$  odd.

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